DIFFERENTIAL GEOMETRY FOR PHYSICISTS

Bo-Yu Hou
Bo-Yuan Hou

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Preface

The developments of physics and mathematics are closely interlinked. In certain respects, mathematics is an important tool for physics, while many significant mathematics problems originate from physics. The relationship between modern theoretical physics and differential geometry is more profound. Physics not only gives new ideas and motivations to modern differential geometry, but also provides powerful alternative methods for geometry. The proof of Atiyah-Singer Index theorem in terms of supersymmetric quantum mechanics\(^1\); Donaldson's analysis of differential structure of \(\mathbb{R}^4\) in terms of the solutions of sourceless non-Abelian gauge field equations\(^2\); and the calculation of Donaldson invariants by Seiberg-Witten in terms of supersymmetric Abelian gauge field with spin\(^3\) are only three of the most prominent examples. On the other hand, in the last few decades, there have been extensive applications of topology and geometry in physics, such as, Riemannian geometry in general relativity\(^4\), Poisson-Lie algebra and symplectic geometry in integrable systems\(^5\), complex geometry and algebraic geometry in conformal field theory and string theory\(^6\), fibre bundle differential geometry in the global analysis of Yang-Mills theory\(^7\) and quantum anomaly\(^8\), etc.. The global and topological analysis gives an impetus to inaugurate many physical idea. All fundamental interactions are gauge interactions, the gauge field potential and gauge field strength identified with the connection and curvature on fibre bundle \(^9\), respectively. In modern quantum field theory and statistical physics, global and nonlinear analysis plays a critical role. The concepts of connection, curvature, characteristic class in differential geometry are showing up in various branches of theoretical physics, and no longer limited to gravity and Yang-Mills theory. For instance, Berry phase\(^{10}\) arises from non-trivial topological properties of parametrical space.

In light of the present situation, this book is written in the hope that it will acquaint our readers and graduate students with some of the basic ideas and methods of modern differential geometry and their applications in physics.

The book is divided into fourteen chapters with another eighteen appendices as introduction to prerequisite topological and algebraic knowledge. The first seven chapters focus on local analysis. This part could be used as an introductional text book for graduate students of theoretical physics. Chapters 8 – 10 discuss geometry on fibre bundle which could be further reference to researchers. The last four chapters deal with the Atiyah-Singer index theorem and its generalization, quantum anomaly, cohomology field theory, and noncommutative geometry, which may give the reader a glimpse of the frontier of current research in theoretical physics.

The first three chapters expound the differential geometry without a metric, in these chapters, three kinds of important differential operators: exterior differential, Lie derivative, and covariant derivative, are introduced. To combine them together,
attention is given to some basic concepts of manifold, mapping, connection and integrability. In chapter one differentiable manifold and tensor field are introduced, and we put emphasis upon the point that tensor fields are geometric entities, independent of the choice of coordinate systems; we emphasize further the coordinate independence of equations. In chapter two, mapping between manifolds and its induced mapping between tensor fields; Lie group manifolds and its invariant vector fields are introduced. In chapter three, we take into consideration additional structures on the manifold: connection and curvature on manifold, though these do not necessarily involve a metric, and analysis of the problem in terms of moving frames are introduced. These are intended to give the readers a visual picture of fibre bundle.

After introducing the connection on manifold, in chapters 4 - 6 we introduce metric structure, symplectic structure, and complex structure on manifold, respectively. In these three chapters, the applications to general relativity, analytical mechanics, symmetric space and integrable systems are also discussed.

Chapter 7 is a critical chapter, from which we start the global analysis on manifold. In chapters 7 and 8, homotopy and homology, which are the fundamental concepts in algebraic topology, are introduced, so are the basic structure and classification of fibre bundles. In chapters 9 and 10, the connection and characteristic class on the fiber bundle are introduced, and we proceed with the global topological analysis in gauge theory. Fibre bundle is the best language for studying the relationship between the global and local properties of manifold.

In chapters 11 - 13, Atiyah-Singer Index theorem, Atiyah-Patodi-Singer Index theorem and family index theorem are introduced briefly. With quantum anomaly as an example, we treat the global analysis in quantum field theory and take account various topological constructions. The last chapter is about non-commutative geometry, quantum group, and q-gauge theory, which are some of the popular topics in current theoretical physics.

At the end of each chapter, a notations and formulae are attached for its summary and reference.

There are many books on differential geometry, and we list some of these which we are more familiar as general references at the end of this book. On the one hand, it is necessary to have an extensive mathematical knowledge, no matter how difficult it is to grasp them immediately for physicists, on the other hand, to the best of our knowledge, these books we known do not go beyond A-S index theorem, and the extension of A-S index theorem (family index theory, index theorem for open and infinity manifold), and their application on quantum field theory are little mentioned. The materials in these areas can only be found in certain articles. Our purpose is to introduce differential geometry, especially the fundamental concept, methods, and results of differential geometry. As for various mathematical terms and definitions, only the most necessary are included, as we never mean to be all-embracing. The straightforward and intuitive approach, which is more acceptable to physicists while not affecting the final mathematical results, has been adopted. References are cited
for the problems discussed, and they are by no means complete, and we apologize to those authors for not mentioning their works. For reader's convenience, the notations and terminology in this book are consistent with the general convention.

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Chapter 1

Differentiable Manifolds and Differential Forms

In Euclidean geometry, two figures are said to be identical if they overlap completely after an Euclidean movement which does not change the Euclidean distance between two points. All of the Euclidean movements form a group. The Euclidean geometry studies the invariant properties of figures under the action of Euclidean movements. From this point of view Klein gave a definition of geometry in 1871 as follows: Given a set $E$ (called space) and a group $G$ which acts on the space $E$, the geometry studies the invariant properties of the space $E$ on which the transformation group $G$ acts. Differential geometry studies the invariant properties of differentiable manifolds under the action of diffeomorphism transformations. The major research objects of differential geometry are differentiable manifolds and various tensor fields on them.

1.1 Manifold

Many problems in physics are related to the continuous spaces, such as the usual space-time in kinetics and dynamics, curved space-time in general relativity, phase space in statistical physics, interior space and related base space(ordinary space-time) in gauge field theory, etc.. These spaces have some common properties: they are all continuous spaces and have a definite dimension. To study them, the concept of manifold is introduced. Manifold is the generalization of point, line, surface and other higher dimensional continuous spaces which we are familiar with.

**Definition 1.1 Manifold:** A real $n$-dimensional manifold $M$ is a space which is like $\mathbb{R}^n$ locally. More precisely, a real $n$-dimensional manifold is such a Hausdorff space in which any point has a neighborhood homeomorphic to $\mathbb{R}^n$.

In the following, we briefly explain some mathematical term in this definition.

1). Real $n$-dimensional linear space $\mathbb{R}^n$. 

1